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INHOMOGENEITY PROBLEM REVISITED VIA THE MODULUS PERTURBATION APPROACH

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Abstract—The present paper reconsiders the inhomogeneity problem via an elastic modulus perturbation approach. By using the eigenstrain concept, the inhomogeneity problem is converted to a series of inclusion problems in the present modulus perturbation procedure. A homogeneous state is chosen as reference state or the 0th order solution. Some extensions of the modulus perturbation approach by example are also discussed in the paper.

1. INTRODUCTION

The investigation of the field of an inhomogeneity embedded in a matrix has long been an attractive subject in micromechanics and the mechanics of composites. However, the definitions of the inclusion and the inhomogeneity have not been unified. In the present investigation the terminology of Mura (1982) for characterizing both inclusion and inhomogeneity will be adopted. The inclusion is defined as a subdomain having eigenstrains prescribed, while the inhomogeneity is defined as a subdomain with a prescribed modulus that is different from the remainder of the material. It is noted that the inhomogeneity can be equivalent to an inclusion with a proper eigenstrain that depends on the external loading and geometry of the body. Unfortunately, the equivalent eigenstrain of an inhomogeneity is too difficult to be obtained for configurations other than that of a single ellipsoidal inhomogeneity embedded in an infinite matrix, as solved by Eshelby (1957). Besides the Eshelby solution, there are only a few other configurations that have analytical solutions. For example, the interaction of two inhomogeneities in an infinite matrix was studied by Moschovidis and Mura (1975), and a semi-spherical inhomogeneity near a free surface has recently been provided by Tsuchida *et al.* (1990). Generally, solutions of this type are too complicated to be used in further applications.

As another branch of micromechanics, the study of the inclusion problem with prescribed eigenstrains has attracted many researchers. Among others, Sankaran and Laird (1976) developed a cuboidal inclusion solution, and Seo and Mura (1979) solved the case of an inclusion near the free surface of a half-space. A direct engineering application of the inclusion solution is found in metallurgical research, such as in studies of phase transformation, where the eigenstrain is known. Also, it is expected that the inclusion solutions may be used for the study of inhomogeneities by the equivalent inclusion method (Mura, 1982, Section 22).

In the present paper, the inhomogeneity problem is formulated via the elastic modulus perturbation approach. It is noted that this procedure (Section 2) turns the inhomogeneity problem to an inclusion problem. The 0th order solution is obtained in a properly selected configuration, say a homogeneous medium, which is chosen to be as simple as possible. The higher order solutions $n \geqslant 1$ show the general properties of an inclusion problem. The corresponding equivalent eigenstrains for $n \geqslant 1$ are obtained from the solutions of previous orders. Through the present perturbation approach, the existing inclusion solutions (Mura, 1982) will have found a new class of applications, which brings additional importance to these solutions. Meanwhile, as it will be pointed out, the perturbation procedure itself, although greatly simplifying the inhomogeneity problem, restricts the solution somewhat. It is easy to see that the difference between C_{ijkl} (matrix modulus) and C^*_{ijkl} (inhomogeneity modulus) is required to be small compared to C_{ijkl} itself. Other remarks on the advantages and disadvantages of the procedure will be made in the last section of the paper.

A similar technique had been used by Walpole (1967) to treat an inclusion in an anisotropic medium. He used an isotropic solution as the reference state, the 0th order solution, which was obtained by Eshelby (1957). Other related works, such as Gao (1991), will be reviewed in the following sections.

2. PERTURBATION PROCEDURE

Consider an elastic medium D with elastic modulus $\mathbf{C}^0$, in which there is a subdomain Ω contained in D with a different modulus $\mathbf{C}^*$. We call Ω the domain of the inhomogeneity and D-Ω as the domain of the matrix. One may write Hooke's law in the matrix as,

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \text{in} \quad D\text{–}\Omega \tag{1a}$$

and in the inhomogeneity as,

$$\sigma_{ij} = C^*_{ijkl}\varepsilon_{kl} \quad \text{in} \quad \Omega. \tag{1b}$$

The perturbation is carried out as follows:

$$\sigma_{ij} = (C^0_{ijkl} + fC^1_{ijkl})\varepsilon_{kl} \quad \text{in} \quad \Omega, \tag{2}$$

where the parameter f may be chosen by the ratio:

$$f = (C^* - C^0)/C^0,$$

where C_0 and C^* are the representative components in C^0_{ijkl} and C^1_{ijkl} respectively.

All the stress and strain fields are expanded as follows:

$$\sigma_{ij} = \sigma^0_{ij} + f\sigma^1_{ij} + O(f^2), \qquad \varepsilon_{ij} = \varepsilon^0_{ij} + f\varepsilon^1_{ij} + O(f^2). \tag{3a, 3b}$$

By identifying terms with the same power of f, the first two order approximations in Ω are:

$$\sigma^0_{ij} = C^0_{ijkl}\varepsilon^0_{kl}. \tag{4a}$$

$O(f)$, 1st order solution:

$$\sigma^1_{ij} = C^0_{ijkl}\varepsilon^1_{kl} + C^1_{ijkl}\varepsilon^0_{kl}. \tag{4b}$$

In general, one has:

$O(f^n)$, nth order solution, $n > 1$:

$$\sigma^n_{ij} = C^0_{ijkl}\varepsilon^n_{kl} + C^1_{ijkl}\varepsilon^{n-1}_{kl}. \tag{4c}$$

On the other hand, in D–Ω,

$$\sigma^n_{ij} = C^0_{ijkl}\varepsilon^n_{kl}, \quad n \geqslant 0. \tag{5}$$

By using the eigenstrain concept (Mura, 1982), eqn (4b) may be rewritten as:

$$\sigma^1_{ij} = C^0_{ijkl}(\varepsilon^1_{kl} - \varepsilon^{1*}_{kl}), \tag{6}$$

where ε^{1*}_{kl} is the so-called eigenstrain, which is obtained by equating (4b) to (6),

$$\boldsymbol{\varepsilon}^{1*} = -(\mathbf{C}^0)^{-1}(\mathbf{C}^1)\boldsymbol{\varepsilon}^0. \tag{7}$$

In terms of the eigenstrain and Green's function of the corresponding domain, the first order approximation is expressed as,

$$u_i^1(\mathbf{x}) = \int_\Omega C^0_{jlmn}\varepsilon^{1*}_{mn}(\mathbf{x}')G_{ij,l}(\mathbf{x},\mathbf{x}')\,\mathrm{d}x' \tag{8a}$$

$$\varepsilon^1_{ij}(\mathbf{x}) = \frac{1}{2}\int_\Omega C^0_{klmn}\varepsilon^{1*}_{mn}(\mathbf{x}')\left(\frac{\partial}{\partial x_j}G_{ik,l}(\mathbf{x},\mathbf{x}') + \frac{\partial}{\partial x_i}G_{jk,l}(\mathbf{x},\mathbf{x}')\right)\mathrm{d}\mathbf{x}', \tag{8b}$$

where $(\ )_{,i} = \partial(\ )/\partial x_i'$.

Green's function $G_{ij}(\mathbf{x},\mathbf{x}')$ in the above equations is the displacement in the x_i direction at point x when the unit point force is acting in the direction x_j at x'. The integrals in eqn (8b) do not exist in the sense of Riemann integrals. A special treatment is needed (see Appendix) when a numerical calculation is carried out.

It is seen that the higher order approximations can be performed in the same fashion. Although in principle the higher order ($n \geqslant 1$) can be obtained, most practical applications only need the first order modification. In the following section, we will mention only the first order formulation most of the time.

Combining eqn (7) with (8a), the perturbation of the displacement equation (8a) may be rewritten as:

$$u_i^1(\mathbf{x}) = -\int_\Omega C^1_{jlmn}\varepsilon^0_{mn}(\mathbf{x}')G_{ij,l}(\mathbf{x},\mathbf{x}')\,\mathrm{d}x' \tag{9a}$$

$$\varepsilon^1_{ij}(\mathbf{x}) = -\frac{1}{2}\int_\Omega C^1_{klmn}\varepsilon^0_{mn}(\mathbf{x}')\left(\frac{\partial}{\partial x_j}G_{ik,l}(\mathbf{x},\mathbf{x}') + \frac{\partial}{\partial x_i}G_{jk,l}(\mathbf{x},\mathbf{x}')\right)\mathrm{d}x'. \tag{9b}$$

Expression (9a) may be obtained by omitting the eigenstrain concept (Gao, 1991). In order to connect the inhomogeneity and inclusion problems, the present analysis shall mention and calculate the eigenstrain as necessary in the following sections.

3. EXAMPLES

In order to have a quantitative sense of the accuracy of the perturbation approximation, the Eshelby (1957) inclusion solution is re-examined. An infinitely long solid cylindrical inhomogeneity is embedded in an isotropic infinite matrix. The far field is loaded by pure shear stress, σ^0_{12}. By using the equivalent inclusion concept (Mura, 1982, Chapter 11), the disturbance due to the inhomogeneity can be expressed as:

$$\sigma_{ij} = C^*_{ijkl}\varepsilon_{kl} = C^0_{ijkl}(\varepsilon_{kl} - \varepsilon^*_{kl}), \tag{10}$$

where σ_{ij} and ε_{ij} are the disturbed fields, and the ε^*_{ij} is the so-called eigenstrain.

From the Eshelby solution, the stresses in the inhomogeneity are known to be uniform. The only non-zero component under shear is σ_{12}. The exact solution of the eigenstrain is:

$$\varepsilon^*_{12(\text{exact})} = \frac{-(\Delta\mu)\varepsilon^0_{12}}{2(\Delta\mu)S_{1212}+\mu}, \tag{11}$$

where $\Delta\mu = \mu^* - \mu$. The constant S_{1212} for the ellipsoidal inclusion was given by Eshelby (1957).

On the other hand, an approximated solution can be obtained from the perturbation procedure given in Section 2 above. The 0th order solution is taken as the uniform shear $\sigma^0_{12} = 2\mu\varepsilon^0_{12}$. In the homogeneity, the first order stress and strain are related as:

$$\begin{pmatrix}\sigma^1_{11}\\ \sigma^1_{22}\\ \sigma^1_{12}\end{pmatrix} = \mathbf{C}^0\begin{pmatrix}\varepsilon^1_{11}\\ \varepsilon^1_{22}\\ \varepsilon^1_{12}\end{pmatrix} + (\mathbf{C}^* - \mathbf{C}^0)\begin{pmatrix}\varepsilon^0_{11}\\ \varepsilon^0_{22}\\ \varepsilon^0_{12}\end{pmatrix}, \tag{12}$$

where

$$\mathbf{C}^0 = 2\mu\begin{bmatrix}\frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0\\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0\\ 0 & 0 & 1\end{bmatrix},\quad \mathbf{C}^* = 2\mu^*\begin{bmatrix}\frac{1-\nu^*}{1-2\nu^*} & \frac{\nu^*}{1-2\nu^*} & 0\\ \frac{\nu^*}{1-2\nu^*} & \frac{1-\nu^*}{1-2\nu^*} & 0\\ 0 & 0 & 1\end{bmatrix} \tag{13}$$

and

$$(\mathbf{C}^0)^{-1} = \frac{1}{2\mu}\begin{pmatrix}1-\nu & -\nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & 1\end{pmatrix}.$$

For simplicity, $\nu = \nu^*$ is taken in the following formulation. The eigenstrain for the first order is obtained by substituting the above formulae into eqn (7):

$$\begin{pmatrix}\varepsilon^{1*}_{11}\\ \varepsilon^{1*}_{22}\\ \varepsilon^{1*}_{12}\end{pmatrix} = -\frac{\Delta\mu}{\mu}\begin{pmatrix}\varepsilon^0_{11}\\ \varepsilon^0_{22}\\ \varepsilon^0_{12}\end{pmatrix}. \tag{14}$$

For the case of pure shear, eqn (14) is the first term of the Taylor expansion of eqn (11). By substituting the eigenstrain into eqn (8b), the first order correction on the strain is obtained. Having noted that the integral (8b) with a uniform eigenstrain has been carried out by Eshelby (1957), ε^1_{ij} can be expressed in terms of the so-called Eshelby tensor S as:

$$\varepsilon^1_{12} = 2S_{1212}\varepsilon^{1*}_{12} = -2S_{1212}\frac{\Delta\mu}{\mu}\varepsilon^0_{12}. \tag{15}$$

In most cases, the 1st order solution provides a good approximation. The error of the present approximation can be obtained by comparing (15) with the exact Eshelby solution.

$$\varepsilon_{12(\mathrm{exact})} = 2S_{1212}\frac{-(\Delta\mu)\varepsilon^0_{12}}{2(\Delta\mu)S_{1212}+\mu} = 2S_{1212}\frac{\Delta\mu}{\mu}\varepsilon^0_{12}\frac{1}{1+2S_{1212}\dfrac{\Delta\mu}{\mu}}.$$

With the Eshelby tensor the discussion can be extended as follows. The next order approximation is read as:

$$\varepsilon^{2*}_{12} = \frac{\Delta\mu}{\mu}\varepsilon^1_{12} = \left(2S_{1212}\frac{\Delta\mu}{\mu}\right)\frac{\Delta\mu}{\mu}\varepsilon^0_{12}, \ldots.$$

By following the same procedure, the nth order eigenstrain is obtained as:

$$\varepsilon_{12}^{n*} = -\left(-2S_{1212}\frac{\Delta\mu}{\mu}\right)^{n-1}\frac{\Delta\mu}{\mu}\varepsilon_{12}^{0}, \tag{16}$$

where $n > 1$.

It is seen that the sum of all the orders of eigenstrains converges to the exact solution, eqn (11), in the range of convergence, i.e.

$$\varepsilon_{12}^{*} = -\frac{\Delta\mu}{\mu}\varepsilon_{12}^{0}\sum_{n=0}^{\infty}\left(-2S_{1212}\frac{\Delta\mu}{\mu}\right)^{n} = \varepsilon_{12(\text{exact})}^{*} \tag{17}$$

if

$$2S_{1212}\frac{\Delta\mu}{\mu} < 1.$$

A numerical range of $\Delta\mu$ can be reached if the shape of the inhomogeneity and Poisson's ratio are given. For a circular cylinder with $\nu = 0.25$, the above condition is read as:

$$\Delta\mu < \tfrac{3}{2}\mu,$$

which will guarantee that the series (17) will converge to the exact solution (11).

Information can be derived from this example concerning the error of the perturbation approximation at the first and higher orders. Usually, only the first order can be easily obtained, while the higher order approximations become more complicated. It is clear that the perturbation solution does not show the extra complexity when the inhomogeneity is not of elliptical shape. The effect of the shape on the stress and strain fields may be discussed by using this approximation scheme. It is noted that this benefit, gained through the perturbation procedure, is at the expense of the restriction that $|\Delta C| = |C^* - C|$ is small.

As another example without a closed form solution, consider an inhomogeneity near a free surface, as shown in Fig. 1, where the inhomogeneity has a radius of 1 and the distance between free surface and the center of the inhomogeneity is h. The homogeneous half-plane is taken again as the 0th order approximation. The first order eigenstrains are easily obtained as the following:

$$\varepsilon_{11}^{0} = \varepsilon_{22}^{0} = \varepsilon_{0}, \quad \varepsilon_{12}^{0} = 0, \tag{18}$$

for biaxial tension loading. The eigenstrains are obtained through

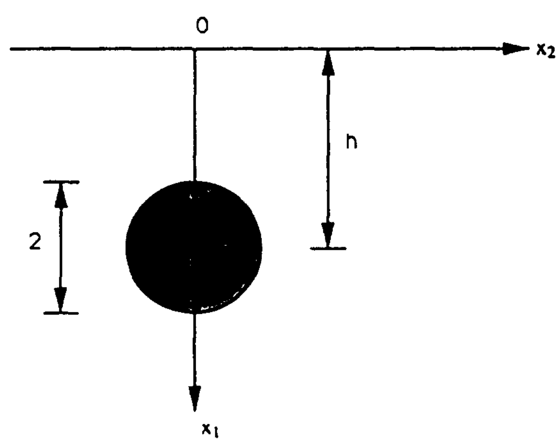

Fig. 1. An inhomogeneity near a surface.

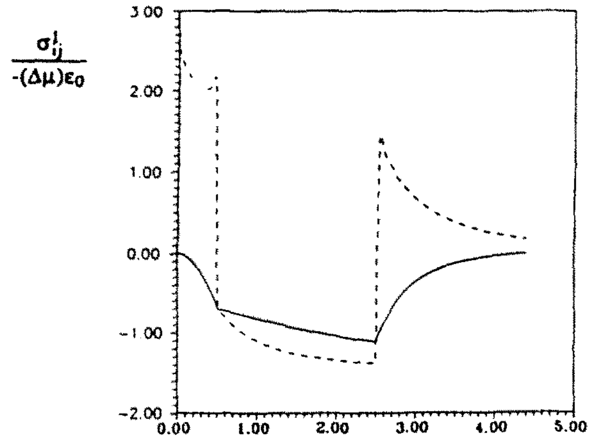

Fig. 2(a). Stress distributions along $x_2 = 0$ with $h = 1.5$ (dashed line is σ_{22}^{I} and solid one is σ_{11}^{I}).

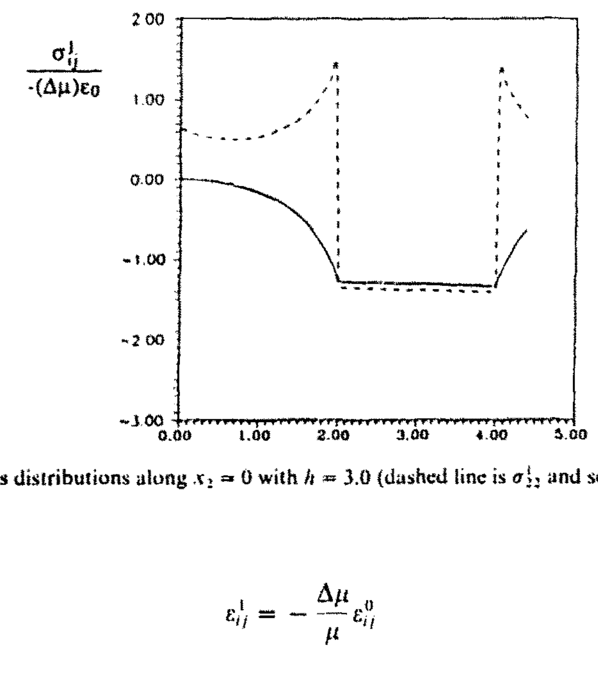

Fig. 2(b). Stress distributions along $x_2 = 0$ with $h = 3.0$ (dashed line is σ_{22}^{I} and solid one is σ_{11}^{I}).

$$\varepsilon_{ij}^{I} = -\frac{\Delta\mu}{\mu}\varepsilon_{ij}^{0} \tag{14}$$

provided that the Poisson's ratios are the same in the homogeneity and matrix.

It is obvious that Green's function used in the integration [eqn (8)] is for a half-plane only. For the sake of completeness, the Green's function is listed in the Appendix of the present paper. Stress distributions for the biaxial case are shown in Fig. 2. It is shown that the Eshelby solution for an inhomogeneity embedded in an infinite space can be used as long as the inhomogeneity is beneath the free surface at roughly twice its radius. Another fact revealed by this calculation is that there is a high tangential stress at the free surface when the inhomogeneity is close to the free surface. This result indicates that fracture failure may first occur at the free surface for a soft inhomogeneity near the surface. On the other hand, the fracture may be observed at the interface of the soft inhomogeneity and the matrix when the inhomogeneity is deeply embedded in the matrix.

4. BOUNDARY CONDITIONS AND MIXED BOUNDARY VALUE PROBLEMS

Beside the perturbation of the governing equations in the inhomogeneity, boundary conditions are also required. It is easy to develop solutions for the stress prescribed condition as:

$$\mathbf{t} = \mathbf{t}^0 \quad \text{on} \quad S_t \tag{19a}$$

and for the displacement prescribed conditions as:

$$\mathbf{u} = \mathbf{u}^0 \quad \text{on} \quad S_u. \tag{19b}$$

The higher order prescribed data are set to zero.

Additional care must be taken when prescribing the boundary conditions for the mixed boundary value problem [see Erdogan (1978)]. It will be noted that the higher order boundary conditions also depend on the previous order solutions. The tractions and displacements in this case must be expanded as infinite series:

$$\mathbf{t} = \mathbf{t}^0 + f\mathbf{t}^1 + \cdots \quad \text{and} \quad \mathbf{u} = \mathbf{u}^0 + f\mathbf{u}^1 + \cdots$$

where $\mathbf{t}^0$ and $\mathbf{u}^0$ are known, while $\mathbf{t}^1$ and $\mathbf{u}^1$ are to be determined from a 0th order solution. In order to clearly present this procedure, the case of a rigid indenter pressed on a semi-infinite plane with a near surface inhomogeneity is considered (as shown in Fig. 3).

By assuming the Hertzian contact area $c \gg d$ (inhomogeneity size), the 0th order boundary conditions are given as:

$$\mathbf{t}^0 = p^0\mathbf{e}_z, \quad \mathbf{u}^0 = u_z^0\mathbf{e}_z. \tag{20}$$

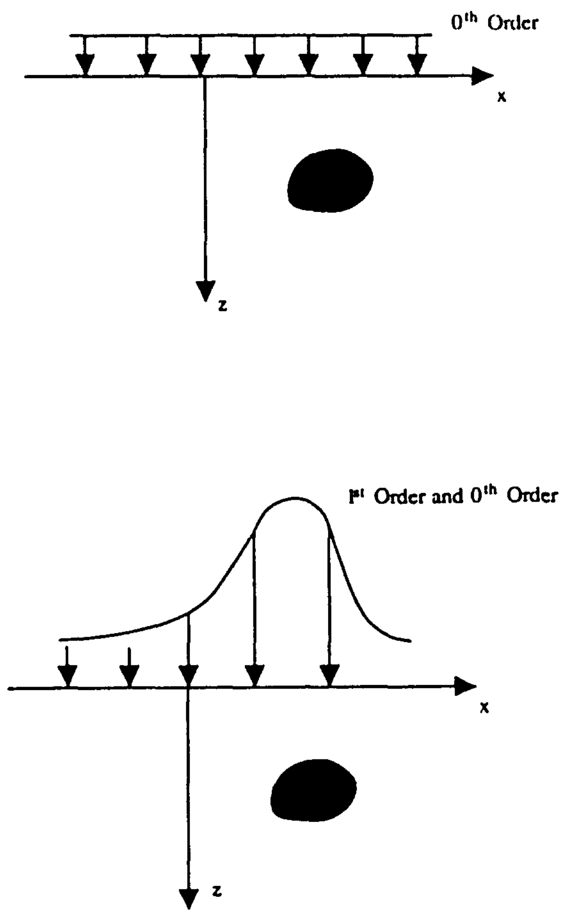

Fig. 3. An inhomogeneity near a contacted region.

They are uniform in the x-direction. The stress and strain fields near the surface are [see Johnson (1985)]:

$$\sigma_x = \sigma_z = p^0, \quad \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \tag{21}$$

and

$$\varepsilon_x = \varepsilon_z = \varepsilon_0, \quad \varepsilon_y = \varepsilon_{xz} = 0. \tag{22}$$

The corresponding eigenstrain is obtained from the equation

$$\begin{pmatrix} \varepsilon_{11}^{1*} \\ \varepsilon_{22}^{1*} \\ \varepsilon_{12}^{1*} \end{pmatrix} = -\frac{\Delta\mu}{\mu}\begin{pmatrix} \varepsilon_0 \\ \varepsilon_0 \\ 0 \end{pmatrix} \tag{23}$$

for $v = v^*$.

From these eigenstrains, the corresponding displacement disturbance at $z = 0$ (barred variables) is given, eqn (8a), as:

$$\begin{aligned} \bar{u}_1^1(\mathbf{x}) = \bar{u}_z^1(\mathbf{x}) &= \int_\Omega C^0_{jkmn}\varepsilon^{1*}_{mn}(\mathbf{x}')G_{lj,k}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' \\ &= -C^1_{jkmn}\varepsilon^0_{mn}\int_\Omega G_{lj,k}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'. \end{aligned} \tag{24}$$

This is a displacement that is not allowed at the contact surface due to the presence of the rigid indenter. Thus, a negative distribution of the above displacement at $z = 0$ is added. The correction to the surface traction at this order is given by the following integral equation:

$$\int_{-L}^{L} \frac{p^1(s)}{x-s}\,\mathrm{d}s = \frac{\pi E}{2(1-v^2)}\frac{\partial \bar{u}_z^1}{\partial x} \tag{25}$$

where $c \gg L \gg d$ is properly selected. The stress traction at $z = 0$ disturbed by the inhomogeneity is expected as in Fig. 3.

The stress or displacement fields of the first order consists of two parts. The first part is computed directly from eqn (8). The second part is obtained by considering $p^1(x)$ acting at $z = 0$ with a homogeneous half plane.

The exact formulation of this problem is complicated because of the interaction between the inhomogeneity and stress distribution within the contact area. Miller and Keer (1983) and Bryant et al. (1984) formulated the problem of two-dimensional contact of an indenter interacting with a near surface inhomogeneity. They provided numerical results for the interaction between a contact indenter at the surface and a near surface circular void or rigid inclusion beneath the indenter. More generally shaped inhomogeneities are too difficult to be solved by an exact formulation. On the contrary, the perturbation approach can treat arbitrarily shaped inhomogeneities, even for the three-dimensional case. The present perturbation scheme has also decoupled the determination of the equivalent eigenstrain of the inhomogeneity and surface traction distribution.

5. OTHER APPLICATIONS VIA THE MODULUS PERTURBATION

The modulus perturbation approach is not limited to the inhomogeneity problem discussed above. Next, two problems are given, which are extensions of the perturbation approach.

Elastic fields in an elastic material of variable modulus

When the elastic modulus is a function of spatial co-ordinates, the problem becomes dramatically more complicated compared to the homogeneous case. For example, Delale and Erdogan (1983) considered a crack in an elastic medium with its modulus changing exponentially as a function of a co-ordinate. It is noted that the perturbation solution can be obtained for this kind of problem under some restrictions.

The 0th order solution is still taken as the solution in the homogeneous body, while the first order equation is written as:

$$\sigma_{ij}^{1} = C_{ijkl}^{0}\varepsilon_{kl}^{1} + C_{ijkl}^{1}(\mathbf{x})\varepsilon_{kl}^{0}, \tag{26}$$

where $C^{1}(\mathbf{x})$ is a known function. The remainder of the formulation is the same as in Section 2 and no special treatment is required for this problem.

A systematic formulation of fracture analysis of nonhomogeneous materials via the moduli perturbation approach was recently provided by Gao (1991). He emphasized the stress intensitiy factor of a crack in the nonhomogeneous elastic medium. Combined with his published works, he has extensively studied the perturbation stress intensity factors of a crack in the elastic medium with $\mathbf{C}(\mathbf{x})$.

Nonlinear elastic and elasto–plastic analysis for high strain hardening materials

As a further extension of the modulus perturbation approach, a nonlinear elastic body [Fig. 4(a)] under a certain loading is considered here. In Fig. 4(a), curve (1) shows a reference material which will be used to construct the 0th order solution. Curve (2) gives the real stress–strain relation. The 0th order solution is the linear elastic solution for a configuration under consideration, while the first order correction is

$$\sigma_{ij}^{1} = C_{ijkl}^{0}\varepsilon_{kl}^{1} + C_{ijkl}^{1}(\sigma_{ij})\varepsilon_{kl}^{0}. \tag{27}$$

If J_2 deformation theory is applied, then one may have $\mathbf{C}^{1} = \mathbf{C}^{1}(J_2)$, where

$$J_2 = \tfrac{3}{2}s_{ij}s_{ij}, \quad s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}. \tag{28}$$

Furthermore an elasto–plastic analysis with a stress–strain curve as in Fig. 4(b) can be treated by the same procedure, although an extra calculation in the 1st order formulation is required to determine the plastic zone and the region where the eigenstrain is prescribed. It is noted that there is (are) some subdomain(s) Ω where the effective stress exceeds the yield limit σ_{ys}. The determination of Ω is achieved according to a yield criterion, e.g. von Mises, as:

$$\tfrac{3}{2}s_{ij}^{0}s_{ij}^{0} \geqslant \sigma_{ys}^{2}, \tag{29}$$

where σ_{ys} is the yield stress in uniaxial tension.

With the above eigenstrain and Ω known, one obtains the first order stress and strain through the integration equation (8). Finally the plastic zone Ω_p of the 1st order where the total effective stress exceeds σ_{ys} is determined by

$$\tfrac{3}{2}(s_{ij}^{0}+fs_{ij}^{1})(s_{ij}^{0}+fs_{ij}^{1}) \geqslant \sigma_{ys}^{2}. \tag{30}$$

It is obvious that higher order perturbations will further correct Ω and the plastic zone Ω_p.

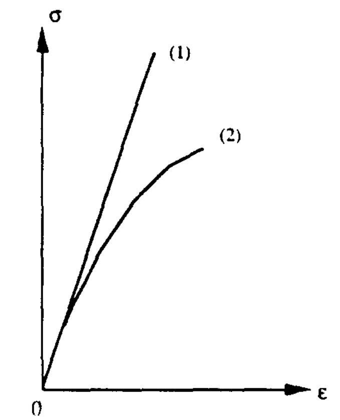

Fig. 4(a). A nonlinear elastic material.

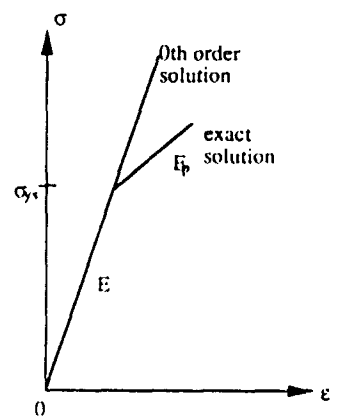

Fig. 4(b). An elasto-plastic material.

6. CONCLUDING REMARKS

We have presented a modulus perturbation scheme for determining the displacement and stress fields disturbed by inhomogeneities. Although FEM or BEM can handle these types of inhomogeneity problems, the present perturbation procedure still shows its advantages by the simplicity and analytical form of the results. By using the modulus perturbation approach, some insight is obtained for many difficult inhomogeneity problems which may be very troublesome from both the computational complexity and the amount of CPU time required in FEM and BEM numerical procedures. For instance, the mixed boundary problem discussed in Section 4, the interaction between a near surface inhomogeneity and surface traction distribution, can be quite costly in CPU time [see a similar calculation on the elastic–plastic contact analysis by Komvopoulos (1989)]. Under the present perturbation procedure, an inhomogeneity problem is converted to a series of inclusion problems. The inhomogeneities other than those of ellipsoidal shape can be considered through the perturbation scheme. Extensions of the application of the modulus perturbation have been made for two examples. The elastic–plastic analysis through the modulus perturbation scheme is easy to be applied to engineering applications.

The perturbation procedure may also be helpful for numerical calculations even for the case when the modulus difference of the inhomogeneity/matrix system is out of the perturbation range. For instance, in an inverse problem [see Gao and Mura (1989)] the initial guess of the solution is vital for the convergence of the numerical iteration scheme. The perturbation solution may provide a reasonable initial guess for the iteration.

It should also be emphasized that there are some limitations on the application of the perturbation scheme. First, the present approach is based on the assumption that ΔC_{ijkl} is small compared to C_{ijkl}. A quantitative restriction is shown in the first example in Section 3, where the exact solution acts as a benchmark. The applicable range of the perturbation solution also depends on the problems and parameters under consideration. Gao (1991) showed that the perturbation solution of stress intensity factor of a Mode III crack surrounded by inhomogeneity is in good agreement with the exact solution even though $\Delta\mu/\mu \geqslant 2$. However, for those problems where ΔC_{ijkl} is small, this assumption may not prove too limiting for engineering applications.

Since the proposed modulus perturbation scheme is a regular perturbation procedure, the approximated solution may lose some important information present in the original problem. For example, as interface crack tip singularity cannot be derived from a 0th order solution which is chosen in a homogeneous space. The singularity at the interface crack tip is a special characteristic of the original problem which must appear in the 0th order configuration. This example illustrates that the application of the perturbation approach should be carried out with great care in terms of the original problem and application.

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APPENDIX: HALF-PLANE GREEN FUNCTION

The Green's function for a half plane can be written as:

$$G_{ij}(\mathbf{x},\mathbf{x}') = G^{k}_{ij}(\mathbf{x},\mathbf{x}') + G^{c}_{ij}(\mathbf{x},\mathbf{x}'), \tag{A1}$$

where $G^{k}_{ij}(\mathbf{x},\mathbf{x}')$ corresponds to the whole-plane Green function and $G^{c}_{ij}(\mathbf{x},\mathbf{x}')$ is called the complementary part of the half-plane Green function. They are given by

$$G^{k}_{ij} = -K_d\left((3-4v)\ln r\delta_{ij} - \frac{r_i r_j}{r^2}\right) \tag{A2}$$

and

$$G^{c}_{11} = K_d\left(-[8(1-v)^2-(3-4v)]\ln R + \frac{(3-4v)R_1^2 - 2c\bar{x}}{R^2} + \frac{4c\bar{x}R_1^2}{R^4}\right).$$

$$G^c_{12} = K_d\left(\frac{(3-4\nu)r_1r_2}{R^2} + \frac{4c\bar{x}R_1r_2}{R^4} - 4(1-\nu)(1-2\nu)\theta\right),$$

$$G^c_{21} = K_d\left(\frac{(3-4\nu)r_1r_2}{R^2} - \frac{4c\bar{x}R_1r_2}{R^4} + 4(1-\nu)(1-2\nu)\theta\right),$$

$$G^c_{22} = K_d\left(-[8(1-\nu)^2-(3-4\nu)]\ln R + \frac{(3-4\nu)r_2^2+2c\bar{x}}{R^2} - \frac{4c\bar{x}r_2^2}{R^4}\right), \tag{A3}$$

where

$$r_i = x_i - x'_i, \quad R_1 = x_1 + x'_1, \quad R_2 = x_2 - x'_2,$$
$$r = (r_ir_i)^{1/2}, \quad R = (R_iR_i)^{1/2}, \quad c = x_1,$$
$$\bar{x} = x'_1, \quad \theta = \arctan\left(\frac{R_2}{R_1}\right), \quad K_d = \frac{1}{8\pi\mu(1-\nu)}.$$

The first order displacement disturbance caused by the inhomogeneity is obtained by using the above Green function and the eigenstrain formulation as follows:

$$u^1_i(\mathbf{x}) = -\int_\Omega C^0_{jlmn}\varepsilon^{1*}_{mn,l}(G^\infty_{ij}(\mathbf{x},\mathbf{x}') + G^c_{ij}(\mathbf{x},\mathbf{x}'))\,d\mathbf{x}'. \tag{A4}$$

Furthermore, the stress disturbance is obtained by taking the derivatives with respect to $\mathbf{x}$, i.e.

$$\sigma^1_{ij} = -\int_\Omega \Sigma_{ijk}(\mathbf{x},\mathbf{x}')C^0_{klmn}\varepsilon^{1*}_{mn,l}(\mathbf{x}')\,d\mathbf{x}' - C^0_{ijmn}\varepsilon^{1*}_{mn}, \tag{A5}$$

where

$$\Sigma_{ijk} = C^0_{ijpq}\left(\frac{\partial}{\partial x_q}G_{pk}(\mathbf{x},\mathbf{x}') + \frac{\partial}{\partial x_p}G_{qk}(\mathbf{x},\mathbf{x}')\right).$$

These equations can also be divided into a whole plane part and a complementary one which is not singular in the domain Ω. These two parts are given as:

$$\Sigma^\infty_{ijk} = -\frac{K_s}{r^2}\left((1-2\nu)(r_k\delta_{ij} + r_j\delta_{ki} - r_i\delta_{jk}) + \frac{2r_ir_jr_k}{r^2}\right) \tag{A6}$$

and

$$\Sigma^c_{111} = -K_s\left(\frac{(3\bar{x}+c)(1-2\nu)}{R^2} + \frac{2R_1(R_1^2+3c\bar{x})-4\bar{x}r_2^2(1-2\nu)}{R^4} - \frac{16c\bar{x}R_1r_2^2}{R^6}\right),$$

$$\Sigma^c_{121} = -K_sr_2\left(-\frac{(1-2\nu)}{R^2} + \frac{2[\bar{x}^2-c^2-2c\bar{x}+2\bar{x}R_1(1-2\nu)]}{R^4} + \frac{16c\bar{x}R_1^2}{R^6}\right),$$

$$\Sigma^c_{221} = -K_s\left(\frac{(\bar{x}+3c)(1-2\nu)}{R^2} + \frac{2[R_1(r_2^2+2c^2)-2cr_2^2+2\bar{x}r_2^2(1-2\nu)]}{R^4} + \frac{16c\bar{x}R_1r_2^2}{R^6}\right),$$

$$\Sigma^c_{112} = -K_sr_2\left(\frac{(1-2\nu)}{R^2} - \frac{2[c^2-\bar{x}^2+6c\bar{x}-2\bar{x}R_1(1-2\nu)]}{R^4} + \frac{16c\bar{x}r_2^2}{R^6}\right),$$

$$\Sigma^c_{122} = -K_s\left(\frac{(3\bar{x}+c)(1-2\nu)}{R^2} + \frac{2[(2c\bar{x}+r_2^2)R_1-2\bar{x}R_1^2(1-2\nu)]}{R^4} - \frac{16c\bar{x}R_1r_2^2}{R^6}\right),$$

$$\Sigma^c_{222} = -K_sr_2\left(\frac{3(1-2\nu)}{R^2} + \frac{2[r_2^2-2c^2-4c\bar{x}-2\bar{x}R_1(1-2\nu)]}{R^4} + \frac{16c\bar{x}R_1^2}{R^6}\right), \tag{A7}$$

where $K_s = 1/4\pi(1-\nu)$.

For a general distribution of eigenstrain, a special treatment on stress evaluation from eqn (A5) is needed (Brebbia *et al.*, 1983, Chapter 6). In the second example of Section 3, the uniformly distributed first order eigenstrain can simplify the present numerical calculation procedure as:

$$\sigma^1_{ij} = -\int_\Omega (\Sigma^\infty_{ijk}(\mathbf{x},\mathbf{x}') + \Sigma^c_{ijk}(\mathbf{x},\mathbf{x}'))C^0_{klmn}\varepsilon^{1*}_{mn,l}(\mathbf{x}')\,d\mathbf{x}'. \tag{A8}$$

Note that the complementary part of the Green function in the above integral is not singular at $\mathbf{x}' = \mathbf{x}$. Thus, the second part of eqn (A8) can be evaluated by using a conventional numerical integration technique. The first part of eqn (A8) is just the Eshelby solution for an inclusion embedded in an infinite matrix which was obtained in a closed form expression.